K22U 2322

Reg. No.: ....

Name : ....

V Semester B.Sc. Degree (CBCSS – OBE – Regular/Supplementary/ Improvement) Examination, November 2022
(2019 Admission Onwards)
CORE COURSE IN MATHEMATICS
5B07MAT : Abstract Algebra

Time : 3 Hours Max. Marks : 48

PART – A

Answer **any 4** questions. They carry **1** mark **each**.

1. Find the order of the cyclic subgroup of $\mathbb{Z}_4$ generated by 3.
2. What is the order of the cycle (1, 4, 5, 7) in S_8 ?
3. Let $\phi : G \to G'$ be a group homomorphism of G onto G′. If G is abelian, prove that G′ is abelian.
4. Let p be a prime. Show that $(a + b)^p = a^p + b^p$ for all $a, b \in \mathbb{Z}_p$.
5. Solve the equation $3x = 2$ in the field $\mathbb{Z}_7$.

PART – B

Answer **any 8** questions from among the questions **6** to **16**. These questions carry **2** marks **each**.

6. Prove that in a group G, the identity element and inverse of each element are unique.
7. Let H and K be subgroups of a group G. Prove that $H \cap K$ is a subgroup of G.
8. State and prove division algorithm for $\mathbb{Z}$.
9. Let G be a group and suppose $a \in G$ generates a cyclic subgroup of order 2 and is the unique such element. Show that $ax = xa$ for all $x \in G$.

P.T.O.

10. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$ be permutations in S
Find $\tau\sigma$ and $|\langle\sigma\rangle|$.

11. Express the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$ in S_8 as a product of disjoint cycles and then as a product of transpositions.

12. Find all orbits of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 6 & 2 & 4 \end{pmatrix}$.

13. Find the index of $\langle 3 \rangle$ in the group of $\mathbb{Z}_{24}$.

14. Prove that every group of prime order is cyclic.

15. Prove that a group homomorphism $\phi : G \to G'$ is a one to one map if and only if $\ker(\phi) = \{e\}$.

16. Let R be a ring with additive identity 0. Then for any $a, b \in R$ prove that

a) $a0 = 0a = 0$

b) $a(-b) = (-a)b = -(ab)$.

PART – C

Answer **any 4** questions from among the questions **17** to **23**. These questions carr
4 marks **each**.

17. Let G be a group and let g be one fixed element of G. Show that the map I_g such that $i_g(x) = gxg'$ for $x \in G$ is an isomorphism of G with itself.

18. Draw subgroup diagram for Klein 4-group V.

19. Let G be a finite cyclic group of order n with generator a. Prove that G is isomorphic to $(\mathbb{Z}_n, +_n)$.

20. Let $n \geq 2$. Prove that the collection of all even permutations of $\{1, 2, 3, \ldots, n\}$ forms a subgroup of order $\frac{n!}{2}$ of the symmetric group S_n.

21. Let H be a subgroup of G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Show that every left coset gH is the same as the right coset Hg.

22. Let H be a subgroup of G. Prove that left coset multiplication is well defined by the equation (aH) (bH) = (ab)H if and only if H is a normal subgroup of G.

23. Let $\phi : \mathbb{Z} \rightarrow S_8$ be homomorphism such that $\phi(1) = (1, 4, 2, 6)\ (2, 5, 7)$. Find ker ($\phi$) and $\phi(20)$.

PART – D

Answer **any 2** questions from among the questions **24** to **27**. These questions carry **6** marks **each**.

24. a) Let G be a cyclic group with n elements and generated by a. Let $b \in G$ and $b = a^s$. Prove that

    i) b generates a cyclic subgroup of H of G containing n/d elements, where d is the gcd of n and s.

    ii) $\langle a^s \rangle = \langle a^t \rangle$ if and only if gcd (s, n) = gcd (t, n).

    b) Let p and q be prime numbers. Find the number of generators of the cyclic group $\mathbb{Z}_{pq}$.

25. a) Prove that every coset (left or right) of a subgroup H of a group G has the same number of elements as H.

    b) State and prove Lagrange's theorem.

26. Let $\phi : G \rightarrow G'$ be a group homomorphism and let H = ker (ϕ). Let $a \in G$. Prove that the set $\phi^{-1}[\{\phi(a)\}] = \{x \in G : \phi(x) = \phi(a)\}$ is the left coset aH of H and is also the right coset Ha of H.

27. a) Prove that every field F is an integral domain.

    b) Prove that every finite integral domain is a field.

    c) Give an example of an integral domain which is not a field.
